

FAILURE OF THE WEIERSTRASS PREPARATION THEOREM IN QUASI-ANALYTIC DENJOY-CARLEMAN RINGS

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ABSTRACT. It is shown that Denjoy-Carleman quasi-analytic rings of germs of functions in two or more variables fail to satisfy the Weierstrass Preparation Theorem. The result is proven via a non-extension theorem.

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1. INTRODUCTION

Denjoy-Carleman rings of infinitely differentiable functions have been classically studied in Partial Differential Equations, Function Theory and other fields of Analysis. Since the mid 1970's, their algebraic structure, which is necessary for understanding the geometry of zero sets of Denjoy-Carleman functions, has also been attracting quite a lot of attention.

The technical complication for Denjoy-Carleman quasi-analytic classes of functions is that applying Weierstrass division to a function in a Denjoy-Carleman class might produce a quotient and a remainder outside that class as shown by Childress [8] in 1976. Since Weierstrass division is the standard method via which the Weierstrass

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Preparation Theorem is proven, the Weierstrass Preparation Theorem became problematic as well as difficult to establish in the wake of Childress' result. Most experts expected it to fail, but a counterexample turned out to be elusive to construct. Partial results on Weierstrass division in the Denjoy-Carleman classes due to Childress, Chaumat, and Chollet gave a glimpse into what a counterexample to the Weierstrass Preparation Theorem needed to be. A function has the Weierstrass division property if it divides any other function in the ring, and the quotient and remainder of that division are inside the ring. A very elementary argument then shows that every function that has the Weierstrass division property satisfies the Weierstrass Preparation Theorem. By Childress' work in 1976 [8] (one direction of the equivalence) and Chaumat-Chollet's work in 2004 [7] (the other direction), a normalized Weierstrass polynomial has the Weierstrass division property in a Denjoy-Carleman ring iff it is hyperbolic, i.e. it only has real roots. This indicated that if the Weierstrass Preparation Theorem failed, it did so on functions that did not have the Weierstrass division property, of which there existed some by Childress' work.

Further complications, however, arise for two reasons. First of all, failure of Weierstrass division seemingly does not imply the failure of the Weierstrass Preparation Theorem, i.e. these two conditions are not equivalent unlike in the classically known cases of holomorphic and real analytic germs. In fact, we are not aware of any example of a ring of smooth functions for which the Weierstrass Preparation Theorem holds, but the Weierstrass division fails. Second of all, standard methods in commutative algebra do not yield any information about the Weierstrass Preparation Theorem.

By contrast, non-quasi-analytic Denjoy-Carleman rings behave much more like rings of smooth functions in the sense that they possess analogs of the Weierstrass (Malgrange) Preparation and Division Theorems as shown in Bronshtein [3]. In establishing such properties of non-quasi-analytic classes, a very important role is played by the fact that functions from these classes can be extended with a *controlled widening* of the class. Various extension results for non-quasi-analytic Denjoy-Carleman classes were proven by Carleson [5], Mityagin [12], Ehrenpreis [9], Wahde [16], Zobin [17, 18] as well as many other authors.

In the case of quasi-analytic classes, however, there are important non-extension results, due to Carleman [4], Lyubich-Tkachenko [11], and, more recently, Langenbruch [10] and Thilliez [15]; see also Nowak [13]. When working on the problems discussed in this article we discovered that the failure of the Weierstrass Preparation Theorem for a quasi-analytic ring (even if we allow any quasi-analytic widening of this ring) follows (and is actually more or less equivalent) to the failure of the extension property for such classes. Unlike these previous results, we prove a different non-extension theorem by producing a very explicit example of non-extendable function with important additional properties that are potent enough to permit contradicting

the Weierstrass Preparation Theorem. Therefore, the main result of this paper is the following:

1.1. Theorem. *The Weierstrass Preparation Theorem does not hold in any quasi-analytic class (strictly containing \mathcal{O}) in dimension ≥ 2 . Moreover, it does not hold even if we allow the unit and the distinguished polynomial to be in any wider quasi-analytic class.*

The main theorem is proven via the following non-extension result:

1.2. Theorem. *Let $M = \{m_n\}$ and $\widetilde{M} = \{\widetilde{m}_n\}$ be two sequences. Then there exists a function f with the following properties:*

- (1) $f \in C^M[0, \infty)$.
- (2) $\forall x \in \mathbb{R}, \forall p \in \mathbb{N} \quad |f^{(p)}(x)| \leq p! 2^{p+1} \widetilde{m}_p$, so in particular, $f \in C^{\widetilde{M}}(\mathbb{R})$.
- (3) *There exists a sequence $\{x_p\} \subset \mathbb{R}$ with $x_p < 0$ and $\lim_{p \rightarrow \infty} x_p = 0$ and a sequence $\{n_p\} \subset \mathbb{N}$, $\lim_{p \rightarrow \infty} n_p = \infty$ such that*

$$\forall p \quad |f^{(n_p)}(x_p)| \geq n_p! \widetilde{m}_{n_p}.$$

- (4) *f is analytic outside of any neighborhood of 0.*

Specifically, this result shows that if we choose a sequence K such that the class $C^K(\mathbb{R})$ is quasi-analytic and $C^K(\mathbb{R}) \subsetneq C^{\widetilde{M}}(\mathbb{R})$, then there exists a function from the quasi-analytic class $C^M[0, \infty)$ which cannot be extended to a function from $C^K(\mathbb{R})$, since a quasi-analytic extension, if it exists, is unique, so it should coincide with this function on the whole axis, and this function does not belong to $C^K(\mathbb{R})$.

It should be noted that the function constructed to satisfy the conclusion of this theorem bears a relationship to the very first example of a Denjoy-Carleman function, which is not real analytic, given by Émile Borel in [1] and [2]; see also [15]. Furthermore, examining the Taylor expansion at 0 of f reveals another important fact. $T_0 f \in \mathcal{F}^M$, so $T_0 f$ is a formal power series satisfying all the derivative bounds given by the sequence M , and yet it corresponds to f , which is not a function in C^M in any neighborhood around the origin as soon as the sequence \widetilde{M} is suitably chosen; see Theorem 3.5 below. Using a functional theoretic argument that produced a lacunary series, Torsten Carleman showed in [4] as early as 1926 that there exist non-convergent formal power series whose coefficients satisfy the Denjoy-Carleman bounds but which do not correspond to Denjoy-Carleman functions in the same class. The function f in Theorem 1.2 finally gives an explicit example of a function of this kind.

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The paper is organized as follows: A preliminary section is devoted to describing in detail the classes of Denjoy-Carleman functions considered as well as the intricacies of Weierstrass division in the Denjoy-Carleman settings and its relation to Weierstrass Preparation. Section 3 deals with extension of functions in quasi-analytic classes. In Section 4, Theorem 1.1 is proven assuming Theorem 1.2. Finally, Section 5 constructs the non-extendable function that proves Theorem 1.2.

2. PRELIMINARIES

Denjoy-Carleman classes – basic definitions and results. The set-up explained below constitutes the most standard one for Denjoy-Carleman classes as detailed in [15].

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will employ the following notation:

$$\begin{aligned} |\alpha| &:= \alpha_1 + \dots + \alpha_n, \\ \alpha! &:= \alpha_1! \dots \alpha_n!, \\ D^\alpha &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \\ x^\alpha &:= x_1^{\alpha_1} \dots x_n^{\alpha_n}. \end{aligned}$$

We also use the following notations:

$\mathcal{E}(U)$ is the space of infinitely smooth complex functions on an open set $U \subset \mathbb{R}^n$,

\mathcal{E}_0 is the ring of germs at the origin of such functions,

\mathcal{F}_n (or simply \mathcal{F}) is the ring of formal power series in n variables,

\mathcal{O}_n (or simply \mathcal{O}) is the ring of convergent power series .

Let $M = \{m_0, m_1, m_2, \dots\}$ be an increasing sequence of positive real numbers with $m_0 = 1$.

2.1. Definition. We say that a function $f \in \mathcal{E}(U)$ belongs to the Denjoy-Carleman class $C_n^M(U)$ if there are constants A, B depending on f , such that

$$\forall x \in U, \forall \alpha \in \mathbb{N}^n \quad |D^\alpha f(x)| \leq |\alpha|! AB^{|\alpha|} m_{|\alpha|}.$$

2.2. Definition. The ring of germs $C_{n,0}^M$ is the inductive limit of $C_n^M(U)$, for U in the family of neighborhoods of the origin.

2.3. Definition. We say that a formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} \frac{a_\alpha}{\alpha!} z^\alpha \in \mathcal{F}_n$$

belongs to the Denjoy-Carleman class \mathcal{F}_n^M if there are constants A, B such that

$$\forall \alpha \in \mathbb{N}^n \quad |a_\alpha| \leq |\alpha|! AB^{|\alpha|} m_{|\alpha|}.$$

In order to get classes of functions with some good structural properties, we must impose some conditions on the sequence $M = \{m_n\}$.

$$(2.1) \quad \frac{m_{j+1}}{m_j} \leq \frac{m_{j+2}}{m_{j+1}} \text{ for all } j \geq 0 \quad (\text{logarithmic convexity}).$$

and

$$(2.2) \quad \sup_j \sqrt[j]{\frac{m_{j+1}}{m_j}} < \infty$$

We summarize the properties of $C_{n,0}^M$ when the sequence M verifies conditions (2.1, 2.2). We omit the n and write simply C_0^M . For the proof and further references, one can consult [15].

2.4. Proposition. *The following properties hold for C_0^M :*

- C_0^M is a local ring containing \mathcal{O} with maximal ideal

$$\mathfrak{m} = \{f \in C_0^M : f(0) = 0\} = (x_1, \dots, x_n)C_0^M.$$

- C_0^M is closed under division by coordinates functions x_1, \dots, x_n .
- C_0^M is closed under composition.
- C_0^M is closed under differentiation.
- The Implicit Function Theorem holds for C_0^M .
- The Inverse Function Theorem holds for $C_{1,0}^M$.

2.5. Definition. A Denjoy-Carleman class C_0^M , properly containing the ring \mathcal{O} , is called *quasi-analytic* if the Taylor map $T_0 : C_0^M \rightarrow \mathcal{F}^M$

$$T_0 f = \sum_{\alpha \in \mathbb{N}^n} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$$

sending each germ to its Taylor series at 0 is injective.

By the Denjoy-Carleman Theorem, the Taylor map is injective if and only if

$$(2.3) \quad \sum_{k=0}^{\infty} \frac{m_k}{(k+1)m_{k+1}} = \infty.$$

Also $C_0^M \supsetneq \mathcal{O}$ if and only if

$$(2.4) \quad \lim_{j \rightarrow \infty} \sqrt[j]{m_j} = \infty.$$

This last condition combined with logarithmic convexity is easily proved to be equivalent to the following one:

$$(2.5) \quad m_{j+1} = \alpha_j m_j \text{ with } \lim_{j \rightarrow \infty} \alpha_j = \infty.$$

We will use both in what follows.

Finally, we recall that the inclusion $C_0^M \subset C_0^{\widetilde{M}}$ is equivalent to the condition

$$\sup_{n \geq 0} \sqrt[n]{\frac{m_n}{\widetilde{m}_n}} < \infty.$$

Basics of Weierstrass Division and Preparation. Assume $n \geq 2$.

We say that $\varphi \in C_{n-1,0}^M[x_n]$, a monic polynomial in x_n of degree d ,

$$\varphi(x) = x_n^d + a_1(x')x_n^{d-1} + \dots + a_d(x'),$$

where $x = (x', x_n)$ and $a_j \in C_{n-1,0}^M$, is a *distinguished Weierstrass polynomial* in x_n if $a_j(0) = 0$, for all $1 \leq j \leq d$.

Such a polynomial φ is called *hyperbolic* in x_n if there exists a neighborhood U of 0 in \mathbb{R}^{n-1} such that $\forall x' \in U$, all the roots of $\varphi(x', \cdot)$ are real; otherwise, φ is called *non-hyperbolic* in x_n .

A germ $f \in C_{n,0}^M$ is *regular in x_n of order d* if there exists a unit u in the ring $C_{n,0}^M$ such that $f(0, x_n) = u(0, x_n) x_n^d$.

Note this is equivalent to saying that

$$f(0, 0) = \frac{\partial}{\partial x_n} f(0, 0) = \dots = \frac{\partial}{\partial x_n^{d-1}} f(0, 0) = 0,$$

while

$$\frac{\partial}{\partial x_n^d} f(0, 0) \neq 0.$$

A germ $f \in C_{n,0}^M$ is *strictly regular in x_n of order d* if

$$\frac{\partial}{\partial x_n^d} f(0, 0) \neq 0, \text{ but for any } \alpha \in \mathbb{N}^n, |\alpha| \leq d-1, \quad D^\alpha f(0, 0) = 0.$$

2.6. Remark. Note that any germ of a smooth function and any formal power series can be made strictly regular with respect to a chosen variable via a linear change of variables.

2.7. Definition. A ring R_n of germs of infinitely differentiable functions in n variables at 0 is said to satisfy the *Weierstrass Preparation Theorem* if for any $d \geq 1$ and any $f \in R_n$, that is regular of order d in x_n , this f can be prepared in R_n , i.e., there exist a unit $u \in \mathbb{R}_n$ and a distinguished Weierstrass polynomial $\varphi \in R_{n-1}[x_n]$ of order d , such that

$$f = u\varphi.$$

2.8. Definition. We say that the *Weierstrass Division Property* holds in $C_{n,0}^M$ for a function f that is regular in x_n of order d if for every $g \in C_{n,0}^M$, there exist $q \in C_{n,0}^M$ and $h \in C_{n-1,0}^M[x_n]$, a polynomial of degree $\leq (d-1)$ in x_n , such that

$$g = fq + h.$$

2.9. Remark. It is classically known that in the ring of holomorphic function germs, the ring of real analytic function germs, and the ring of smooth germs, the Weierstrass Preparation Theorem holds true and they also have the Weierstrass Division Property.

2.10. Remark. What are the relations between the two Weierstrass properties?

The Weierstrass Division Property for an element f regular in x_n of order d in a ring of germs R_0 implies that the same f can be prepared. Indeed, it is enough to divide x_n^d by f , and we obtain $x_n^d = qf + r$ with $q, r \in R_0$. Comparing orders we see that q must be a unit and the remainder r is a polynomial in x_n of degree $d-1$ whose coefficients must vanish at 0. Hence $f = q^{-1}(x_n^d - r)$ is prepared.

We do not know if the Weierstrass Preparation Theorem for a ring R_0 implies the Weierstrass Division Property for this ring. A natural way to establish this hits an obstacle: assume the Weierstrass Preparation Theorem holds true in R_0 and take $f, g \in R_0$. We can assume both are regular in the same variable, so that there exist units u, v and Weierstrass polynomials p, q such that $f = up$, $g = vq$. Perform the Euclidean division between the two Weierstrass polynomials p and q , we get $p = aq + r$ and thus $f = up = v^{-1}(ua)vq + ur = v^{-1}uag + ur$. There is no guarantee, however, that ur can be reduced to a polynomial.

From Childress [8] and Chaumat-Chollet [7], we know that for $n \geq 2$ a Weierstrass polynomial $g \in C_{n,0}^M$ has the Weierstrass Division Property if and only if it is a hyperbolic polynomial. In particular, the Weierstrass Division Property does not hold in general in $C_{n,0}^M$ (the reader can find specific examples of functions for which it fails in [8] and [15]), but this fact does not imply the failure of the Weierstrass Preparation Theorem.

Note that the Weierstrass Division Property, and, hence, the Weierstrass Preparation, both hold in the ring \mathcal{F}^M , and, moreover, this ring is Noetherian. This was proved by Chaumat et Chollet in [6]. Yet, the Taylor morphism $C_0^M \rightarrow \mathcal{F}^M$ is not surjective as proven long ago by Carleman in [4], so the better algebraic properties of \mathcal{F}^M do not automatically transfer to C_0^M .

2.11. *Remark.* What about the Weierstrass Division and Preparation in *non-quasi-analytic* Denjoy-Carleman rings $C_{n,0}^{\{m_k\}}$? M. Bronshtein [3] has shown that the Weierstrass Division Property holds *weakly*: for any $d \in \mathbb{N}$, for every $f \in C_{n,0}^{\{m_k\}}$, regular of order d in x_n , and for every function $g \in C_{n,0}^{\{m_k\}}$, there exist a function q from a *wider* Denjoy-Carleman ring $C_{n,0}^{\{m_{dk}\}}$ and a polynomial $h \in C_{n-1,0}^{\{m_{dk}\}}[x_n]$ over this wider ring, such that

$$g = fq + h.$$

From this result, he deduced that, given $d \in \mathbb{N}$, every $f \in C_{n,0}^{\{m_k\}}$ regular of order d in x_n can be prepared in this wider ring $C_{n,0}^{\{m_{dk}\}}$, meaning that there exist a unit $u \in C_{n,0}^{\{m_{dk}\}}$ and a distinguished Weierstrass polynomial $\varphi \in C_{n-1,0}^{\{m_{dk}\}}[x_n]$ of degree d , such that

$$f = u\varphi.$$

So a *controlled* widening of the ring (depending upon the regularity of f) saves the day for both Weierstrass theorems. This widening is optimal.

This result suggests that in the non-quasi-analytic case one should consider a wider ring,

$$\tilde{C}^M(U) = \bigcup_{d \in \mathbb{N}} C^{\{m_{kd}\}}(U).$$

In this new class, the Weierstrass Division Theorem holds and therefore every function can be prepared.

When we started working on the quasi-analytic case, we expected to find a wider quasi-analytic ring containing all the objects we were looking for. Instead, it turned out that the situation is drastically different in the quasi-analytic case.

Our main result asserts:

Let $C_{n,0}^M$, $n \geq 2$ be a quasi-analytic Denjoy-Carleman ring. Choose ANY wider quasi-analytic Denjoy-Carleman ring $\tilde{C}_{n,0}^M$. There exists $f \in C_{n,0}^M$, regular of order 2 in x_n , which cannot be prepared in $\tilde{C}_{n,0}^M$.

So we say that the Weierstrass Preparation Theorem *strongly fails* in quasi-analytic Denjoy-Carleman rings in dimensions ≥ 2 .

3. EXTENSION OF FUNCTIONS FROM DENJOY-CARLEMAN CLASSES

Given a quasi-analytic local ring $C_0^M \supsetneq \mathcal{O}$, we will need to find a wider ring \tilde{C}_0^M verifying two conditions pulling in opposite directions. On one hand, we require \tilde{C}_0^M to be quasianalytic, that is to verify 2.3. On the other hand, we ask that

$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{\tilde{m}_n}{m_n}} = \infty$, which in particular implies $C_0^M \subsetneq \tilde{C}_0^M$. This is done in the following lemma:

3.1. Lemma. *For any quasi-analytic local ring C_0^M there exists a quasi-analytic local ring $C_0^{\widetilde{M}}$ with $C_0^M \subset C_0^{\widetilde{M}}$ and $\lim_{j \rightarrow \infty} \sqrt[j]{\frac{\widetilde{m}_j}{m_j}} = \infty$.*

Proof. Since C_0^M is quasi-analytic, we have $\sum_{j \geq 1} \frac{m_{j-1}}{j \cdot m_j} = \infty$. Put $\alpha_j = \frac{m_j}{m_{j-1}}$ and

$$\mu_j = \sum_{k=1}^j \frac{1}{k \alpha_k}. \text{ So } \lim_{j \rightarrow \infty} \mu_j = \infty. \text{ Define } \beta_j = \alpha_j \sqrt{\mu_j}, \text{ so that } \lim_{j \rightarrow \infty} \frac{\beta_j}{\alpha_j} = \infty.$$

Then $\sum \frac{1}{j \beta_j}$ diverges. Indeed, its partial sums verify

$$\sum_{k=1}^j \frac{1}{k \beta_k} = \sum_{k=1}^j \frac{1}{k \alpha_k \sqrt{\mu_k}} > \frac{1}{\sqrt{\mu_j}} \sum_{k=1}^j \frac{1}{k \alpha_k} = \sqrt{\mu_j}.$$

Now let us define $\widetilde{m}_j = \beta_j \widetilde{m}_{j-1}$, $\widetilde{m}_0 = 1$. We obtain that

- $C_0^{\widetilde{M}}$ is quasi-analytic by construction.
- $\frac{\widetilde{m}_j}{m_j} = \frac{\beta_j \widetilde{m}_{j-1}}{\alpha_j m_{j-1}}$, hence $\lim_{j \rightarrow \infty} \sqrt[j]{\frac{\widetilde{m}_j}{m_j}} = \infty$ by the equivalence of conditions (2.4) and (2.5), applied to \widetilde{m}_j/m_j and β_j/α_j .
- The previous limit implies $\sup_{j \geq 0} \left(\frac{m_j}{\widetilde{m}_j} \right)^{\frac{1}{j}} < \infty$ as needed for $C_0^M \subset C_0^{\widetilde{M}}$.

□

3.2. Definition. For $U = [0, \varepsilon) \subset \mathbb{R}_+$ let

$$C_{+,0}^M(U) = \{f \in \mathcal{E}(U) : \exists A, B > 0 \quad \forall k \in \mathbb{N}, \forall x \in U \quad |D^k f(x)| \leq k! A B^k m_k\}.$$

The space $C_{+,0}^M$ is the space of germs at 0 of such functions.

It is very natural to ask whether every germ in $C_{+,0}^M$ is the restriction of some germ in $C_{1,0}^M$, i.e., if

$$C_{+,0}^M \subset C_{1,0}^M|_{\mathbb{R}_+},$$

or, more generally, if

$$(3.1) \quad C_{+,0}^M \subset C_{1,0}^{\widetilde{M}}|_{\mathbb{R}_+}$$

for a wider class $C_{1,0}^{\widetilde{M}}$, with \widetilde{M} depending only upon M .

If this is true, we say, respectively, that $C_{+,0}^M$ has the *strong extension property* or the *weak extension property*.

3.3. Remark. For a non-quasi-analytic class $C_{+,0}^M$ there are various descriptions of (generally, wider) classes $C_{1,0}^{\widetilde{M}}$ satisfying (3.1); see for example the papers mentioned

in the introduction. So the weak (but usually not the strong) extension property holds for non-quasi-analytic Denjoy-Carleman classes.

The situation of quasi-analytic Denjoy-Carleman classes is quite different. The crucial difference from the previous case is that if a germ from a quasi-analytic class $C_{+,0}^M$ extends to a germ from another quasi-analytic class $C_{1,0}^{\widetilde{M}}$, then this extension is unique due to the quasi-analyticity. In fact, this uniqueness of extension often prevents the existence of an extension.

In this framework, there are interesting results by Langenbruch [10] and Thilliez [14]; see also Nowak [13]. There one can find the following statement:

3.4. Theorem. *Let $C_{1,0}^M$ be a quasi-analytic local ring. If $C_{1,0}^M$ properly contains the ring of analytic germs \mathcal{O}_1 and $C_{1,0}^M \subset C_{1,0}^{\widetilde{M}}$, then $C_{+,0}^M \setminus (C_{1,0}^{\widetilde{M}}|_{\mathbb{R}_+}) \neq \emptyset$.*

So Theorem 3.4 asserts that *any* quasi-analytic local ring $C_{1,0}^M$ containing \mathcal{O} fails the weak extension property, i.e., no matter how wide is the quasi-analytic class $C_{1,0}^{\widetilde{M}}$ we choose, there still exist germs in $C_{+,0}^M$, which do not extend to germs from $C_{1,0}^{\widetilde{M}}$.

In the last section, we reprove this result by a very explicit construction of non-extendible germs, which have useful additional properties that we plan to use in our further research. Our construction yields another result not implied by Theorem 3.4 that we need in the next section. Namely, the following is an easy consequence of Theorem 1.2:

3.5. Theorem. *Let $C_{1,0}^M, C_{1,0}^N$ be quasi-analytic Denjoy-Carleman rings. Assume that $C_{1,0}^M$ properly contains \mathcal{O}_1 . Then there exists a quasi-analytic ring $C_{0,1}^K$ such that $C_{1,0}^N \subset C_{1,0}^K$, and*

$$(C_{+,0}^M \cap C_{1,0}^K|_{\mathbb{R}_+}) \setminus C_{1,0}^N|_{\mathbb{R}_+} \neq \emptyset.$$

Proof. First, let us note that, due to Lemma 3.1, there exists a sequence $K = \{k_p\}$, such that $C_{0,1}^K$ is a quasi-analytic ring, $C_{1,0}^N \subset C_{1,0}^K$ and

$$(3.2) \quad \lim_{j \rightarrow \infty} \left(\frac{k_j}{n_j} \right)^{\frac{1}{j}} = \infty.$$

Next, let f be a function as in Theorem 1.2 for $\widetilde{M} = K$. Let us verify that

$$f|_{\mathbb{R}_+} \in (C_{+,0}^M \cap C_{1,0}^K|_{\mathbb{R}_+}) \setminus C_{1,0}^N|_{\mathbb{R}_+}.$$

According to Theorem 1.2, $f \in C_{1,0}^K$, and $f|_{\mathbb{R}_+} \in C_{+,0}^M$. Let us show that $f \notin C_{1,0}^N$, which will prove the Theorem, since the restriction map $C_{1,0}^K \rightarrow C_{1,0}^K|_{\mathbb{R}_+}$ is injective.

Assume the opposite, $f \in C_{1,0}^N$. Then there exist $A, B > 0$ such that $|f^{(p)}(x)| \leq p!AB^pn_p$ for all $p \in \mathbb{N}$ and for all points x in a neighborhood of 0.

By Theorem 1.2 we have

$$|f^{(p_j)}(x_j)| \geq p_j! k_{p_j}.$$

Therefore we get

$$\forall j \quad p_j! k_{p_j} \leq p_j! AB^{p_j} n_{p_j}.$$

So for any j

$$\sqrt[p_j]{\frac{k_{p_j}}{n_{p_j}}} \leq A^{\frac{1}{p_j}} B,$$

which contradicts (3.2). □

4. A STRONG FAILURE OF THE WEIERSTRASS PREPARATION THEOREM

4.1. Theorem. *For any quasi-analytic local ring $C_{n,0}^M$, $n \geq 2$, and for any wider local quasi-analytic ring $C_{n,0}^{\widetilde{M}}$ there exists $g \in C_{n,0}^M$, regular of order 2, which cannot be prepared in $C_{n,0}^{\widetilde{M}}$.*

Proof. It suffices to consider the case of dimension 2.

Choose a quasi-analytic ring $C_{1,0}^K \supsetneq C_{n,0}^{\widetilde{M}}$ as in Lemma 3.1. By Theorem 3.5 we find a function f on a neighborhood $U \subset \mathbb{R}$ of 0, such that

$$f|_{U \cap \mathbb{R}_+} \in C^M(U \cap \mathbb{R}_+), \quad f \in C^K(U), \quad \text{but } f \notin C^{\widetilde{M}}(U).$$

We can assume that $f(0) = 0$ and $f'(0) > 0$. If this is not the case, it suffices to take $f(x) + kx + b$, which is in the same class as f and has the desired property for suitable constants k and b .

In particular, f is strictly increasing in a neighborhood W of 0, so it is invertible on this neighborhood, and it maps W onto a neighborhood V of 0. Also, because f is increasing on W and $f(0) = 0$, it maps $W \cap \mathbb{R}_+$ onto $V \cap \mathbb{R}_+$.

From Proposition 2.4, we deduce that

$$f^{-1}|_{V \cap \mathbb{R}_+} \in C^M(V \cap \mathbb{R}_+), \quad f^{-1}|_V \in C^K(V) \quad \text{but } f^{-1}|_V \notin C^{\widetilde{M}}(V).$$

Define $g(x, t) = f(t^2) - x$. Since $t^2 \geq 0$, then, by Proposition 2.4, $g(x, t)$ is a $C_{2,0}^M$ germ. Since $f'(0) \neq 0$, $g(t, x)$ is regular of order 2 with respect to the variable t .

Let us show that g cannot be prepared in $C_{2,0}^{\widetilde{M}}$.

Assuming the opposite, we find neighborhoods Y, Z of the origin in \mathbb{R} , a second order (with respect to t) Weierstrass polynomial $P(t, x) \in C^{\widetilde{M}}(Z)[t]$ and a non-vanishing function $Q(t, x) \in C^{\widetilde{M}}(Y \times Z)$, such that

$$\forall (t, x) \in Y \times Z \quad g(t, x) = P(t, x)Q(t, x).$$

Without loss of generality (contracting W , and therefore V , if necessary), we may assume that

$$Y = \{t \in \mathbb{R} : t^2 \in W \cap \mathbb{R}_+\}, \quad Z = V.$$

P does not have terms of degree 1 in t because there are no terms of degree 1 in t in T_0g . Therefore,

$$P(t, x) = t^2 - a(x), \quad a \in C^{\widetilde{M}}(V).$$

Since g vanishes on the curve $x = f(t^2)$ in $Y \times Z$, and Q does not vanish in $Y \times Z$, we see that $P(t, x) = t^2 - a(x)$ vanishes on this curve in $Y \times Z$. Conversely, P vanishes on the curve $t^2 = a(x)$ in $Y \times Z$, therefore g also vanishes on this curve. This means that the curves $x = f(t^2)$ and $t^2 = a(x)$ coincide in $Y \times Z$.

All points $x \in V \cap \mathbb{R}_+ = Z \cap \mathbb{R}_+$ are of the form $x = f(t^2)$ for some $t \in W \cap \mathbb{R}_+$. Thus

$$\forall x \in V \cap \mathbb{R}_+ \quad x = f(a(x)).$$

It follows that

$$\forall x \in V \cap \mathbb{R}_+ \quad f^{-1}(x) = a(x).$$

The condition $f^{-1}|_{V \cap \mathbb{R}_+} \in C^M(V \cap \mathbb{R}_+)$ implies $a|_{V \cap \mathbb{R}_+} \in C^M(V \cap \mathbb{R}_+)$. By our assumptions, however, $a \in C^{\widetilde{M}}(V) \subset C^K(V)$. Also, $f^{-1}|_V \in C^K(V)$. Since $C^K(V)$ is a quasi-analytic class, $f^{-1} - a \in C^K(V)$, and $f^{-1} - a$ is identically zero on $V \cap \mathbb{R}_+$. Hence $f^{-1} = a$ on all of V . So

$$f^{-1} \in C^{\widetilde{M}}(V),$$

which means, again by Proposition 2.4, that

$$f \in C_{1,0}^{\widetilde{M}},$$

contradicting the construction of f . □

5. AN EXPLICIT NON-EXTENDABLE FUNCTION

In this section, we prove Theorem 1.2 by constructing an explicit example of a function in question.

Proof. Our bricks for constructing the function f will be functions

$$g_n = \frac{A_n}{2^n(z_n - x)},$$

where $z_n = x_n + iy_n$ with $x_n < 0$, $y_n > 0$, $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$, and A_n suitably chosen.

Step 1. We can choose A_n in such a way that $g = \sum g_n \in C^{\widetilde{M}}(\mathbb{R})$.

Proof. It is enough to show that

$$|g_n^{(p)}(x)| = p! \frac{A_n}{2^n |z_n - x|^{p+1}} \leq p! 2^{-n} \tilde{m}_p.$$

To do this, consider a function $\tilde{\varphi}$ related to one of the classic ways of gauging the growth of the sequence $\{\tilde{m}_t\}$:

$$\tilde{\varphi}(\xi) = \sup_{t \geq 0} \frac{\xi^{t+1}}{\tilde{m}_t}$$

We have the following:

- The function $\tilde{\varphi}(\xi)$ is increasing for $\xi > 0$ since it is the supremum of monotonically increasing functions.
- For each $\xi > 0$ we get $\tilde{\varphi}(\xi) < \infty$, $\lim_{\xi \rightarrow \infty} \tilde{\varphi}(\xi) = \infty$ and $\xi^{t+1} \leq \tilde{\varphi}(\xi) \cdot \tilde{m}_t$.

Now for any $x \in \mathbb{R}$, we have

$$\left| \frac{A_n}{(z_n - x)^{p+1}} \right| \leq \frac{|A_n|}{|y_n|^{p+1}}$$

Taking $A_n = \frac{1}{\tilde{\varphi}(y_n^{-1})}$ yields

$$|g_n^{(p)}(x)| \leq p! \frac{1}{2^n} \frac{(y_n^{-1})^{p+1}}{\tilde{\varphi}(y_n^{-1})} \leq p! \frac{1}{2^n} \tilde{m}_p.$$

Hence we obtain

$$|g^{(p)}(x)| \leq p! \tilde{m}_p$$

□

Step 2. A suitable choice of $\{x_n\}$ implies $g \in C^M[0, \infty)$.

Proof. Observe that for $x \geq 0$, we have

$$\left| \frac{A_n}{(z_n - x)^{p+1}} \right| \leq \frac{|A_n|}{|x_n|^{p+1}} = \frac{|x_n^{-1}|^{p+1}}{\tilde{\varphi}(y_n^{-1})}.$$

Now define a function $\varphi(\xi)$ with respect to the sequence m_n in the same way as we defined $\tilde{\varphi}(\xi)$ by setting

$$\varphi(\xi) = \sup_{t \geq 0} \frac{\xi^{t+1}}{m_t}.$$

It enjoys the same properties as $\tilde{\varphi}$. In particular, both are injective for $\xi > 0$ as they are monotonically increasing, so we can take \bar{x}_n such that

$$\varphi(\bar{x}_n^{-1}) = \tilde{\varphi}(y_n^{-1}).$$

Taking $x_n = -\bar{x}_n$, we have

$$\begin{aligned}
|g^{(p)}(x)| &\leq p! \sum_n |g_n^{(p)}(x)| \leq p! \sum_n \frac{1}{2^n} \cdot \frac{|A_n|}{|x_n|^{p+1}} = p! \sum_n \frac{1}{2^n} \cdot \frac{|x_n^{-1}|^{p+1}}{\tilde{\varphi}(y_n^{-1})} = \\
&= p! \sum_n \frac{1}{2^n} \frac{|x_n^{-1}|^{p+1}}{\varphi(|x_n|^{-1})} \leq p! m_p.
\end{aligned}$$

□

In order to have a function f verifying property (3), we have to extract a suitable subsequence $\{g_{n_j}\} \subset \{g_n\}$ and we must be careful in choosing the sequence $\{y_n\}$.

Let $\{n_j\}$ any subsequence of \mathbb{N} . We shall estimate the n_j^{th} derivative of the function $S = \sum_t g_{n_t}$ at the point x_{n_j} as follows:

$$\begin{aligned}
(5.1) \quad |S^{(n_j)}(x_{n_j})| &= \left| g_{n_j}^{(n_j)}(x_{n_j}) + \sum_{t \neq j} g_{n_t}^{(n_j)}(x_{n_j}) \right| \\
&\geq \left| g_{n_j}^{(n_j)}(x_{n_j}) \right| - \sum_{t \neq j} |g_{n_t}^{(n_j)}(x_{n_j})|.
\end{aligned}$$

On the right side of the inequality above, there are three terms, namely

$$\begin{aligned}
a) \quad & \left| g_{n_j}^{(n_j)}(x_{n_j}) \right| = n_j! \frac{(y_{n_j}^{-1})^{n_j+1}}{2^{n_j} \tilde{\varphi}(y_{n_j}^{-1})} \\
b) \quad & \sum_{t < j} |g_{n_t}^{(n_j)}(x_{n_j})| \\
c) \quad & \sum_{t > j} |g_{n_t}^{(n_j)}(x_{n_j})|
\end{aligned}$$

We shall evaluate them in subsequent steps.

Step 3. Call $b_n = \frac{\tilde{m}_n}{\tilde{m}_{n-1}}$. Then for all $z \in [b_n, b_{n+1}]$, one has

$$\tilde{m}_n = \sup_{\xi > 0} \frac{\xi^{n+1}}{\tilde{\varphi}(\xi)} = \frac{z^{n+1}}{\tilde{\varphi}(z)}.$$

Proof. a) $\forall j \leq n \quad \frac{z^{j-1}}{\tilde{m}_{j-1}} \leq \frac{z^j}{\tilde{m}_j}.$

Indeed, $\frac{z^j}{\tilde{m}_j} = \frac{z^{j-1}}{\tilde{m}_{j-1}} \cdot \frac{z}{b_j} \geq \frac{z^{j-1}}{\tilde{m}_{j-1}}$ since $z \geq b_n \geq b_j$.

b) $\forall j \geq n \quad \frac{z^{j+1}}{\tilde{m}_{j+1}} \leq \frac{z^j}{\tilde{m}_j}.$

Indeed, $\frac{z^{j+1}}{\tilde{m}_{j+1}} = \frac{z^j}{\tilde{m}_j} \cdot \frac{z}{b_{j+1}} \leq \frac{z^j}{\tilde{m}_j}$ since $b_{j+1} \geq b_{n+1} \geq z$.

Thus

$$\forall z \in [b_n, b_{n+1}] \quad \tilde{\varphi}(z) = \sup_{k>0} \frac{z^{k+1}}{\tilde{m}_k} = \frac{z^{n+1}}{\tilde{m}_n}.$$

From the above, it follows that

$$\frac{z^{n+1}}{\tilde{\varphi}(z)} = \tilde{m}_n \geq \sup_{\xi>0} \frac{\xi^{n+1}}{\tilde{\varphi}(\xi)} \geq \frac{z^{n+1}}{\tilde{\varphi}(z)}$$

because $\xi^{t+1} \leq \tilde{\varphi}(\xi)\tilde{m}_t$ for all $\xi > 0, t > 0$.

Consequently, these inequalities are equalities. \square

Step 4. Choosing $y_n^{-1} \in [b_n, b_{n+1}]$, we have

$$|g_n^{(n)}(x_n)| = 2^{-n}n! \tilde{m}_n \quad \forall n \in \mathbb{N}$$

Proof. $|g_n^{(n)}(x_n)| = n! \frac{(y_n^{-1})^{n+1}}{2^n \tilde{\varphi}(y_n^{-1})} = 2^{-n}n! \tilde{m}_n.$

\square

Step 5. $\forall l \exists N > l$ such that $\forall t < l$ and $\forall n > N$ the following inequality holds

$$\frac{(y_t^{-1})^{n+1}}{\tilde{\varphi}(y_t^{-1})} < 2^{-(n+2)} \tilde{m}_n$$

Proof.

$$\frac{(y_t^{-1})^{n+1}}{\tilde{\varphi}(y_t^{-1})} = \frac{(y_t^{-1})^{t+1}}{\tilde{\varphi}(y_t^{-1})} \cdot (y_t^{-1})^{n-t} = \tilde{m}_t (y_t^{-1})^{n-t} = \tilde{m}_n \frac{y_t^{-1}}{b_{t+1}} \cdot \frac{y_t^{-1}}{b_{t+2}} \cdots \frac{y_t^{-1}}{b_n} \leq \tilde{m}_n 2^{-(n+2)}.$$

The second equality comes directly from Step 3.

For the last inequality, take N' such that $b_{N'} > 4b_{t+1}$. Note that with this assumption the factors up to the $(N' - 1)^{th}$ one are all bounded above by 1, while the rest are bounded above by $1/4$. Since there are $n - N' + 1$ of the latter factors, we obtain the upper bound $(1/4)^{n-N'+1}$.

Choosing $n > N = 2N'$, we obtain that $N' < n/2$, and

$$\frac{1}{4^{n-N'+1}} < \frac{1}{4^{(n/2)+1}} = \frac{1}{2^{n+2}}.$$

\square

Step 6.

$$|S^{(n_j)}(x_{n_j})| \geq n_j! \left[2^{-n_j} \tilde{m}_{n_j} - \max_{t < j} \frac{(y_{n_t}^{-1})^{n_j+1}}{\tilde{\varphi}(y_{n_t}^{-1})} - 2^{1-n_{j+1}} \tilde{m}_{n_j} \right]$$

Proof. From equation (5.1), we have

$$\begin{aligned}
|S^{(n_j)}(x_{n_j})| &\geq |g_{n_j}^{(n_j)}(x_{n_j})| - \sum_{t \neq j} |g_{n_t}^{(n_j)}(x_{n_j})| \geq \\
&\geq |g_{n_j}^{(n_j)}(x_{n_j})| - \sum_{t < j} |g_{n_t}^{(n_j)}(x_{n_j})| - \sum_{t > j} |g_{n_t}^{(n_j)}(x_{n_j})| \geq \\
&\geq n_j! \left[\frac{(y_{n_j}^{-1})^{n_j+1}}{2^{n_j} \tilde{\varphi}(y_{n_j}^{-1})} - \max_{t < j} \frac{(y_{n_t}^{-1})^{n_j+1}}{\tilde{\varphi}(y_{n_t}^{-1})} - 2^{1-n_{j+1}} \max_{\xi > 0} \frac{\xi^{n_j+1}}{\tilde{\varphi}(\xi)} \right] = \\
&= n_j! \left[2^{-n_j} \tilde{m}_{n_j} - \max_{t < j} \frac{(y_{n_t}^{-1})^{n_j+1}}{\tilde{\varphi}(y_{n_t}^{-1})} - 2^{1-n_{j+1}} \tilde{m}_{n_j} \right]
\end{aligned}$$

□

Step 7. Finally, we define the sequence $\{n_j\}$ recursively. Suppose we have already defined n_{j-1} . Define n_j as N in Step 5 with $l = n_{j-1}$, also making sure that

$$n_j - n_{j-1} \geq 3.$$

Step 8. Now we have

$$\max_{t < j} \frac{(y_{n_t}^{-1})^{n_j+1}}{\tilde{\varphi}(y_{n_t}^{-1})} \leq 2^{-(n_j+2)} \tilde{m}_{n_j}$$

and

$$2^{1-n_{j+1}} \tilde{m}_{n_j} \leq 2^{-2-n_j} \tilde{m}_{n_j}.$$

Putting it all together, we obtain the inequality

$$|S^{(n_j)}(x_{n_j})| \geq n_j! 2^{-n_j} \left(1 - \frac{1}{4} - \frac{1}{4} \right) \tilde{m}_{n_j} = \frac{1}{2} n_j! 2^{-n_j} \tilde{m}_{n_j}$$

for all j .

Since our construction always gives

$$\lim_{n \rightarrow \infty} A_n = 0 \text{ and } \lim_{n \rightarrow \infty} x_n = 0,$$

we have uniform convergence of the series $\sum g_n(x)$ on a complex neighborhood of any set $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$, which guarantees the analyticity of $f(x) = 2S(2x)$ on any such set, proving that this function satisfies all properties in Theorem 1.2. □

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